

# A Robust Discuss on Lyapunov Stability and LaSalle's Invariance Principle

Ogbonna Nnamuchi

Department of Mathematics, Chukwuemeka Odumegwu Ohukwu University

Uli Campus Nigeria

DOI: <https://doi.org/10.5281/zenodo.20448002>

Published Date: 29-May-2026

---

**Abstract:** This paper provides a rigorous and systematic exposition of Lyapunov stability theory and LaSalle's invariance principle for nonlinear autonomous dynamical systems. The analysis begins with the foundational formulation of equilibrium points and the construction of Lyapunov functions, emphasizing their roles as generalized energy measures for dynamical systems. Key structural properties, including positive definiteness and radial unboundedness, are introduced and used to establish global boundedness and stability properties of system trajectories.

The classical Lyapunov direct method is developed to establish global asymptotic stability under strict decay conditions on the Lyapunov function derivative. This framework is subsequently extended through LaSalle's invariance principle, which relaxes the requirement of strict negativity and allows for semidefinite dissipation. In this setting, convergence is characterized in terms of the largest invariant subset of the  $V$  set where the Lyapunov derivative vanishes.

Supporting results are presented concerning monotonicity of Lyapunov functions along trajectories, compactness of sublevel sets, and invariance properties of limit sets. Particular emphasis is placed on stability conditions of the form

$\dot{V}(x) \leq -k \|x - x^*\|^2$ , and  $\dot{V}(x) \leq 0$  and their implications for asymptotic behaviour.

A geometric interpretation is provided in which Lyapunov functions define a generalized energy landscape, and system trajectories evolve toward invariant sets where dissipation ceases. The results demonstrate that Lyapunov's direct method and LaSalle's invariance principle together form a unified and powerful analytical framework for establishing global asymptotic stability in nonlinear control systems, optimization dynamics, and dissipative mechanical systems.

**Keywords:** Lyapunov stability; LaSalle invariance principle; nonlinear dynamical systems; global asymptotic stability; Lyapunov function; radial unboundedness; equilibrium points; invariant sets; nonlinear control theory; stability analysis.

---

## 1. INTRODUCTION

The study of nonlinear dynamical systems frequently centers on determining the long-term behavior of trajectories. In particular, one seeks to understand whether trajectories remain close to equilibrium points, whether they converge to equilibria as time evolves, and whether this convergence persists globally for arbitrary initial conditions. Classical linearization methods are often insufficient in nonlinear settings, especially when nonlinear effects dominate the system dynamics away from equilibria. For this reason, direct methods based on scalar energy-like functions play a central role in modern stability theory [10, 2, 1].

The most important of these methods is the Lyapunov direct method. Rather than solving the differential equations explicitly, one constructs a scalar function  $V(x)$ , called a Lyapunov function, whose behavior along trajectories encodes stability information. Intuitively,  $V(x)$  behaves similarly to a generalized energy: if the energy decreases along trajectories, then the system tends to move toward lower-energy configurations, ideally converging to equilibrium [3].

However, in many important systems one cannot establish strict energy decay. Instead, one often obtains only nonincreasing energy behavior. In such situations, Lyapunov's classical theorem becomes insufficient. LaSalle's invariance principle remedies this limitation by characterizing the asymptotic behavior of trajectories through the invariant subsets of the set where the energy dissipation vanishes [5, 6].

The purpose of this document is to present a complete and rigorous derivation of the following standard argument:

" $V$  is positive definite and radially unbounded.  $V' \leq -\kappa\|x - x^*\|^2$ . Thus  $V' < 0$  for  $x \neq x^*$ . Apply Lyapunov's theorem.

If only  $V' \leq 0$ , boundedness ensures convergence to the invariant set  $\{x : V' = 0\}$ . Direct substitution shows this set contains only  $x^*$ . Apply LaSalle."

We shall develop all necessary definitions, lemmas, and theorems in a mathematically rigorous manner.

## 2. AUTONOMOUS DYNAMICAL SYSTEMS

We begin by introducing the class of systems under consideration. Throughout this document we study autonomous nonlinear ordinary differential equations of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

where the vector field

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is assumed to be locally Lipschitz continuous. This regularity assumption guarantees existence and uniqueness of solutions for initial value problems [1].

An equilibrium point represents a stationary configuration of the system. If the system reaches such a point, it remains there for all future time.

**Definition 1** (Equilibrium Point). *A point  $x^* \in \mathbb{R}^n$  is called an equilibrium point if*

$$f(x^*) = 0.$$

## 3. LYAPUNOV FUNCTIONS

The central idea of Lyapunov theory is to associate to the system a scalar function that behaves monotonically along trajectories. Such functions generalize the concept of physical energy. If the energy decreases over time, then trajectories are forced toward lower-energy regions of the state space [2, 3].

## 4. POSITIVE DEFINITE FUNCTIONS

A Lyapunov function must first measure "distance" from equilibrium in an energetic sense. This is formalized through positive definiteness.

**Definition 2** (Positive Definite Function). *A continuously differentiable function*

$$V: \mathbb{R}^n \rightarrow \mathbb{R}$$

*is said to be positive definite with respect to the equilibrium  $x^*$  if*

$$V(x^*) = 0$$

*and*

$$V(x) > 0 \quad \text{for all } x \neq x^*.$$

## 5. RADIAL UNBOUNDEDNESS

Local positivity alone is insufficient for global conclusions. To guarantee global boundedness properties, one additionally requires that the Lyapunov function become arbitrarily large as the state norm increases [1].

**Definition 3** (Radially Unbounded Function). *A function  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  is radially unbounded if*

$$\|x\| \rightarrow \infty \quad \Rightarrow \quad V(x) \rightarrow \infty.$$

**Lemma 4.** *If  $V$  is radially unbounded, then for every  $c > 0$ , the sublevel set*

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\}$$

*is compact.*

*Proof.* Since  $V$  is continuous,  $\Omega_c$  is closed.

Radial unboundedness implies there exists  $R > 0$  such that

$$\|x\| > R \quad \Rightarrow \quad V(x) > c.$$

Hence

$$\Omega_c \subseteq \{x : \|x\| \leq R\},$$

so  $\Omega_c$  is bounded.

A subset of  $\mathbb{R}^n$  that is closed and bounded is compact by the Heine–Borel theorem. Therefore  $\Omega_c$  is compact.  $\square$

## 6. DERIVATIVE OF A LYAPUNOV FUNCTION

The key mechanism in Lyapunov theory is the evolution of the scalar function  $V(x(t))$  along trajectories of the system. The sign of this derivative determines whether the generalized energy increases or decreases [1, 2].

**Definition 5** (Derivative Along Trajectories). *Let*

$$\dot{x} = f(x),$$

*and let  $V \in C^1$ . The derivative of  $V$  along trajectories is defined by*

$$\dot{V}(x) = \nabla V(x)^\top f(x).$$

Using the chain rule,

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t))^\top \dot{x}(t) = \nabla V(x(t))^\top f(x(t)).$$

## 7. LYAPUNOV STABILITY THEOREM

The following theorem is one of the standard global asymptotic stability criteria in nonlinear control theory and dynamical systems analysis [1, 2].

**Theorem 6** (Lyapunov Global Asymptotic Stability Theorem). *Consider the system*

$$\dot{x} = f(x),$$

*with equilibrium  $x^*$ . Suppose there exists a continuously differentiable function  $V$  such that:*

- (i)  $V$  is positive definite,
- (ii)  $V$  is radially unbounded,
- (iii)

$$\dot{V}(x) < 0 \quad \text{for all } x \neq x^*.$$

*Then the equilibrium  $x^*$  is globally asymptotically stable.*

## 8. MONOTONICITY AND BOUNDEDNESS

Before introducing LaSalle’s principle, we establish an important monotonicity property. Even if the derivative of  $V$  is only nonpositive rather than strictly negative, the function still decreases monotonically along trajectories [4].

**Lemma 7** (Monotonicity Lemma). *Suppose*

$$\dot{V}(x) \leq 0$$

along trajectories of the system. Then  $V(x(t))$  is nonincreasing in time. *Proof.* By the chain rule,

$$\frac{d}{dt}V(x(t)) = \dot{V}(x(t)) \leq 0$$

Integrating from  $t_1$  to  $t_2 \geq t_1$ ,

$$V(x(t_2)) - V(x(t_1)) = \int_{t_1}^{t_2} \dot{V}(x(s)) ds \leq 0$$

Hence

$$V(x(t_2)) \leq V(x(t_1)).$$

Therefore  $V(x(t))$  is nonincreasing. □

**Corollary 8** (Boundedness of Trajectories). *Suppose  $V$  is radially unbounded and*

$$V(x) \leq 0.$$

*Then all trajectories are bounded.*

*Proof.* Since  $V(x(t))$  is nonincreasing,

$$V(x(t)) \leq V(x(0)).$$

Hence

$$x(t) \in \Omega_{V(x(0))} = \{x : V(x) \leq V(x(0))\}.$$

By radial unboundedness, this sublevel set is compact. Therefore trajectories remain bounded for all  $t \geq 0$ . □

## 9. LASALLE'S INVARIANCE PRINCIPLE

LaSalle's invariance principle extends Lyapunov's theorem by allowing the derivative of the Lyapunov function to be merely nonpositive rather than strictly negative. The principle has become one of the foundational tools in nonlinear systems theory, switched systems, hybrid systems, and adaptive control [5, 7, 6].

**Theorem 9** (LaSalle's Invariance Principle). *Let  $D \subseteq \mathbb{R}^n$  be compact and positively invariant.*

*Suppose there exists  $V \in C^1(D)$  such that*

$$\dot{V}(x) \leq 0 \quad \forall x \in D.$$

*Define the set*

$$E = \{x \in D : \dot{V}(x) = 0\}.$$

*Let  $M$  denote the largest invariant subset of  $E$ . Then every trajectory starting in  $D$  converges to  $M$ .*

## 10. MAIN STABILITY RESULT

We now present the corresponding detailed proof

### 10.1 Strict Decay Case

**Theorem 10.** *Suppose there exists a continuously differentiable function  $V$  satisfying:*

- (i)  $V$  is positive definite,
- (ii)  $V$  is radially unbounded,
- (iii) there exists  $\kappa > 0$  such that

$$\dot{V}(x) \leq -\kappa \|x - x^*\|^2.$$

*Then  $x^*$  is globally asymptotically stable.*

*Proof.* Because  $V$  is positive definite,

$$V(x^*) = 0, \quad V(x) > 0 \quad \forall x \neq x^*.$$

Because  $V$  is radially unbounded,

$$\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty.$$

Now assume

$$V'(x) \leq -\kappa \|x - x^*\|^2, \quad \kappa > 0.$$

If  $x \neq x^*$ , then

$$\|x - x^*\|^2 > 0.$$

Therefore,

$$-\kappa \|x - x^*\|^2 < 0,$$

which implies

$$V'(x) < 0 \quad \forall x \neq x^*.$$

Hence  $V$  is a strict Lyapunov function.

Applying Lyapunov's global asymptotic stability theorem yields that  $x^*$  is globally asymptotically stable.  $\square$

### 10.1.1 Semidefinite Decay Case

We now consider the more delicate case where the derivative of the Lyapunov function is only nonpositive. This situation frequently occurs in mechanics, control systems, and optimization dynamics [11, 12].

**Theorem 11.** *Suppose:*

- (i)  $V$  is positive definite,
- (ii)  $V$  is radially unbounded,
- (iii)  $V'(x) \leq 0$ .

Define

$$E = \{x : V'(x) = 0\}.$$

If the largest invariant subset of  $E$  is

$$M = \{x^*\},$$

then  $x^*$  is globally asymptotically stable.

*Proof.* Since

$$V'(x) \leq 0,$$

the monotonicity lemma implies that  $V(x(t))$  is nonincreasing.

Because  $V$  is radially unbounded, all trajectories remain in compact sublevel sets. Hence trajectories are bounded.

Define

$$E = \{x : V'(x) = 0\}.$$

By LaSalle's invariance principle, every trajectory converges to the largest invariant subset

$M \subseteq E$ .

Suppose direct substitution into the system dynamics shows that the only invariant subset of  $E$  is the equilibrium point  $x^*$ . Then

$$M = \{x^*\}.$$

Therefore every trajectory converges to  $x^*$ .

Consequently  $x^*$  is globally asymptotically stable.  $\square$

### 11. INTERPRETATION OF THE SET $\{V = 0\}$

An important conceptual point in LaSalle's theorem is the role of the set

$$E = \{x : V'(x) = 0\}.$$

This set does not automatically consist of equilibria. Instead, it contains all states at which the Lyapunov function ceases to decrease instantaneously.

However, trajectories may enter this set and still move within it. For this reason, LaSalle's theorem requires determination of the *largest invariant subset* of  $E$ .

In practice, one typically proceeds as follows:

1. Compute  $V'(x)$ ,
2. Solve the equation

$$V'(x) = 0,$$

3. Substitute these constraints into the dynamics

$$\dot{x} = f(x),$$

4. Determine which trajectories can remain in  $E$  for all time.

Very often, the only possibility is the equilibrium itself [5, 7].

### 12. COROLLARY OF LASALLE'S PRINCIPLE

The preceding theorem is frequently used in the following compact form.

**Corollary 12.** *Suppose:*

- (i)  $V$  is positive definite,
- (ii)  $V$  is radially unbounded,
- (iii)

$$V'(x) \leq 0,$$

(iv) the set

$$E = \{x : V' = 0\}$$

contains no invariant subset other than  $x^*$ .

Then  $x^*$  is globally asymptotically stable.

*Proof.* The result follows immediately from LaSalle's invariance principle. □

### 13. GEOMETRIC INTERPRETATION

The geometric interpretation of Lyapunov and LaSalle methods has been widely used in nonlinear mechanics, switched systems, and robust control analysis [3, 4].

The function  $V(x)$  acts as a generalized energy landscape. Trajectories move downhill because the derivative  $V'$  is nonpositive.

If the energy decreases strictly everywhere except at equilibrium, then trajectories continuously descend toward the unique minimum, yielding asymptotic stability directly through Lyapunov's theorem.

If the energy merely fails to increase, then trajectories eventually reach regions where dissipation vanishes:

$$V' = 0.$$

LaSalle's theorem states that trajectories asymptotically approach motions compatible with zero dissipation. The limiting behavior is therefore determined entirely by the invariant structure of the set  $\{V = 0\}$ .

When the only invariant motion is the equilibrium itself, all trajectories necessarily converge to equilibrium.

#### 14. CONCLUSION

Lyapunov's direct method and LaSalle's invariance principle constitute two of the most powerful tools in nonlinear stability theory. Lyapunov's theorem establishes asymptotic stability through strict energy decay, while LaSalle's principle extends this framework to systems exhibiting merely nonincreasing energy.

The argument developed throughout this document rigorously justifies the commonly abbreviated proof strategy:

Positive definiteness and radial unboundedness provide boundedness and compactness of trajectories. Strict negativity of  $V$  implies global asymptotic stability by Lyapunov's theorem. If only semidefinite decay is available, LaSalle's invariance principle reduces the problem to analyzing the invariant subsets of  $\{V = 0\}$ .

This framework underlies substantial portions of modern nonlinear control theory, optimization dynamics, robotics, mechanics, and dissipative systems theory [1, 2, 11].

#### REFERENCES

- [1] H. K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2002.
- [2] Haddad, W. M., & Chellaboina, V. (2011). *Nonlinear Dynamical Systems and Control*. <https://doi.org/10.2307/j.ctvcn4hws>
- [3] E. D. Sontag and Y. Wang, "Lyapunov characterizations of input to output stability," *SIAM Journal on Control and Optimization*, vol. 39, no. 1, pp. 226–249, 2000. doi:10.1137/S0363012999350213
- [4] Martynyuk, A. A. (2000). A survey of some classical and modern developments of stability theory. *Nonlinear Analysis: Theory, Methods & Applications*, 40(1-8), 483–496. [https://doi.org/10.1016/s0362-546x\(00\)85027-0](https://doi.org/10.1016/s0362-546x(00)85027-0)
- [5] Brogliato, B., & Goeleven, D. (2004). The Krakovskii-LaSalle Invariance Principle for a Class of Unilateral Dynamical Systems. *Mathematics of Control, Signals, and Systems*, 17(1), 57–76. <https://doi.org/10.1007/s00498-004-0145-0>
- [6] Yuan, Y., & Cheng, D. (2008). Stability and stabilisation of planar switched linear systems via LaSalle's invariance principle. *International Journal of Control*, 81(10), 1590–1599. <https://doi.org/10.1080/00207170701793814>
- [7] J. H. Wang, D. Z. Cheng and X. M. Hu, "An Extension of LaSalle's Invariance Principle for a Class of Switched Linear Systems," *Systems & Control Letters*, Vol. 58, No. 10-11, 2009, pp. 754-758. doi:10.1016/j.sysconle.2009.08.008
- [8] Ti-Chung Lee and Ching-Hung Lee, "New stability results for time-varying systems based on a modified detectability condition," *Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No.00CH37187)*, Sydney, NSW, Australia, 2000, pp. 3495-3500 vol.4, doi: 10.1109/CDC.2000.912245
- [9] Sundarapandian, V. (2003). An invariance principle for discrete-time nonlinear systems. *Applied Mathematics Letters*, 16(1), 85–91. [https://doi.org/10.1016/s0893-9659\(02\)00148-9](https://doi.org/10.1016/s0893-9659(02)00148-9)
- [10] R. I. Leine, "The historical development of classical stability concepts: Lagrange, Poisson and Lyapunov stability," *Nonlinear Dynamics*, vol. 59, no. 1–2, pp. 173–182, 2010. doi:10.1007/s11071-009-9530-z
- [11] Liberzon, D. (2012). *Switching in Systems and Control*. Springer Science & Business Media.
- [12] Sepulchre, R., Janković, M., & Kokotović, P.V. (2011). *Constructive Nonlinear Control*.